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ON THE ROTATION OF A ROTLET OR SPHERE IN THE PRESENCE OF AN INTERFACE

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Abstract—The flows produced by a rotlet and a sphere rotating perpendicular to an interface separating two immiscible fluids are studied, and exact solutions are constructed in terms of the stream function and velocity. A comprehensive set of values for the torque acting on a sphere for various values of the parameters defining the ratio of sphere radius to its distance from the interface and the ratio of viscosities is given.

INTRODUCTION

The motion of a small particle which is near or at the fluid–fluid interface between two stratified immiscible fluids is of considerable interest in chemical engineering science because of its relevance to understanding the transport mechanics of filtration and the hydrodynamics of monomolecular surfactant layers. The general problem poses considerable mathematical difficulties as well as conceptual difficulties arising from the relevant physical processes because of the change in shape of the interface. This is particularly apparent when a particle straddles an interface and has a translational motion perpendicular to it. In this case consideration of the effect of the moving contact line must be taken into account. It is therefore not surprising that there are only three solutions for this type of problem available in the literature. First is the pure rotation of a sphere straddling and steadily rotating about a diameter perpendicular to an interface. This solution is given by Schneider *et al.* (1973) assuming a perfectly flat interface throughout the motion. Second is the solution of Bart (1968) for the motion of a viscous fluid which moves normal to a flat and non-deforming interface. In this work, a condition of zero normal velocity at the interface replaces the condition of continuity of normal velocity. The third solution is that of Ranger (1978) where a circular disc lying in the interface and instantaneously moving normal to it, is considered. Here the continuity conditions on the velocity and stress are adhered to, but the fluid motion is assumed to be quasi-steady Stokes flow with a deforming interface which instantaneously is assumed to be flat.

The problem considered by Schneider *et al.* has the advantages that the motion is steady with a velocity parallel to the interface, and that the neglect of any departure of the shape of the interface from exactly planar does not lead to any great errors. This has been verified by the experiments of Kunesh (1971) when one of the fluid phases is air. With these advantages in mind, our aim in this paper is to present a study of the axisymmetric rotation of a rotlet or solid sphere near a planar fluid–fluid interface. A rotlet can be of use in modelling an axisymmetric particle of arbitrary shape when it is sufficiently far away from the interface, and the solution to this problem is constructed in terms of a function proportional to the velocity. In this case, the motion of the fluids is truly steady as there is no motion perpendicular to the interface. The more general problem of a sphere rotating in the presence of an interface is solved by use of bispherical coordinates. The expression for the torque acting on the sphere is in the form of an infinite series and a comprehensive set of numerical values are given for its variation with the parameters defining the ratio of the sphere radius to its distance from the interface, and the ratio of viscosities.

AXISYMMETRIC ROTATION IN STOKES FLOW

If the motion is slow the fluid velocity **q** can be written as

$$\mathbf{q} = \frac{V(x,\rho)}{\rho}\hat{\phi}, \qquad [1]$$

where (x, ρ, ϕ) are cylindrical polar coordinates and $\hat{\phi}$ is the unit vector pendicular to the azimuthal plane ϕ = constant, and in the sense of ϕ increasing, $V(x, \rho)$ is a solution of

$$L_{-1}(V) \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho}\right)V = 0. \qquad [2]$$

Consider the solution of [2] expressed by

$$V = \frac{\sin^2\theta}{r} = \frac{\rho^2}{r^3}, \qquad [3]$$

where $x = r\cos\theta$, $\rho = r\sin\theta$, define spherical polar coordinates.

The surfaces V = constant correspond to the streamlines for an axisymmetric dipole located at the origin. The torque over a sphere typified by r = constant, is given by

$$\mathbf{G} = \int_0^\pi \int_0^{2\pi} [\mathbf{r}_\wedge \mathbf{R}_r] r^2 \sin\theta \, d\theta \, d\phi \qquad [4]$$

where the stress vector $\mathbf{R}_r$ is

$$\begin{aligned} r\mathbf{R}_r &= -p\mathbf{r} + \mu\{(\mathbf{r}\cdot\nabla)\mathbf{q} - \mathbf{q} + \nabla(\mathbf{q}-\mathbf{r})\} \\ &= -p\mathbf{r} + \mu\left(r\frac{\partial}{\partial r} - 1\right)\frac{V}{r\sin\theta}\hat{\phi}, \end{aligned} \qquad [5]$$

p being the pressure and μ the viscosity. If [5] is substituted in [4] it is found that

$$\mathbf{G} = 8\pi\mu\hat{k} \qquad [6]$$

so that the torque is independent of r and is constant over any sphere concentric with the origin. The motion produced by [3] is that due to an axisymmetric rotlet located at the origin.

Consider now a two phase flow of immiscible fluids in which the interface is defined by $x = 0$ and there is no normal component of velocity on the interface. The viscosity of the fluid for $x > 0$ is μ_1 and μ_2 for $x < 0$. The motion is produced by an axisymmetric rotlet located at $x = 1$, $\rho = 0$. Appropriate forms for the velocity in the two regions $x \gtrless 0$ are given by

$$V_1 = \frac{\rho^2}{R_1^3} + \frac{A\rho^2}{R_2^3}, \qquad x > 0, \qquad [7]$$

$$V_2 = \frac{B\rho^2}{R_1^3}, \qquad x < 0, \qquad [8]$$

where $\mathbf{q}_i = (V_i/\rho)\hat{\phi}$, $R_1^2 = (x-1)^2 + \rho^2$, $R_2^2 = (x+1)^2 + \rho^2$ and A, B are constants. It is readily verified that V_1 and V_2 are solutions of [2] and A, B are to be determined by conditions at the interface.

Now the interface conditions require that the velocity is continuous on $x = 0$, that is

$$V_1 = V_2 \quad \text{at } x = 0 \qquad [9]$$

which yields

$$1 + A = B. \qquad [10]$$

The normal and tangential components of stress are also continuous at the interface. The former condition is automatically satisfied and the latter can be expressed as

$$\mu_1 \frac{\partial V_1}{\partial x} = \mu_2 \frac{\partial V_2}{\partial x} \quad \text{at } x = 0. \qquad [11]$$

This gives rise to

$$\lambda(1 - A) = B, \qquad \lambda = \frac{\mu_1}{\mu_2}. \qquad [12]$$

Solving [10] and [12] determines A and B as follows:

$$A = \frac{\lambda - 1}{\lambda + 1}, \qquad B = \frac{2\lambda}{1 + \lambda} \qquad [13]$$

so that

$$V_1 = \frac{\rho^2}{R_1^3} + \left(\frac{\lambda - 1}{\lambda + 1}\right)\frac{\rho^2}{R_2^3}, \qquad x > 0 \qquad [14]$$

$$V_2 = \frac{2\lambda}{1 + \lambda}\frac{\rho^2}{R_1^3}, \qquad x < 0. \qquad [15]$$

Since $R_2 \geq R_1$ in $x \geq 0$ the velocities are both positive everywhere, and on the interface

$$V_1 = V_2 = \frac{2\lambda\rho^2}{(1 + \lambda)(1 + \rho^2)^{3/2}} \qquad [16]$$

which is uniformly small when λ is small or μ_2 is large compared with μ_1. In the limit $\lambda \to 0$, the solution for a rotlet in the presence of a plane is obtained. The analysis presented in this section is easily generalized to the situation of an additional rotlet in $x < 0$. As $R_1, R_2 \to \infty$

$$V_1 = V_2 \sim \frac{2\lambda}{1 + \lambda}\frac{\rho^2}{R_1^3}. \qquad [17]$$

SPHERE ROTATING PERPENDICULAR TO AN INTERFACE

In this section the rotlet will be replaced by a sphere whose centre is fixed and is rotating about the axis of symmetry with angular velocity ω. Again the fluid velocity $\mathbf{q}_j$ $(j = 1, 2)$ is represented by

$$\mathbf{q}_j = \frac{V_j}{\rho}\hat{\phi} \qquad [18]$$

where $L_{-1}(V_j) = 0$, $j = 1, 2$. The boundary and interface conditions are expressed by

$$\left.\begin{array}{ll} V_1 = \omega\rho^2 & \text{on sphere} \\ V_1 = V_2 & \text{at } x = 0 \\ \mu_1 \dfrac{\partial V_1}{\partial x} = \mu_2 \dfrac{\partial V_2}{\partial x} & \text{at } x = 0 \end{array}\right\}. \qquad [19]$$

If bispherical coordinates are defined by

$$\rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \quad x = \frac{c \sinh \xi}{\cosh \xi - \cos \eta} \tag{20}$$

then the sphere is typified by $\xi = \alpha$, and suitable solutions for V_j, $j = 1, 2$ are given by

$$V_1 = \omega c^2 (\cosh \xi - \cos \eta)^{-1/2} \times \sum_{n=1}^{\infty} \left[A_n \cosh\left(n + \frac{1}{2}\right)\xi + B_n \sinh\left(n + \frac{1}{2}\right)\xi \right] P_n^1(\cos \eta) \sin \eta \tag{21}$$

$$V_2 = \omega c^2 (\cosh \xi - \cos \eta)^{-1/2} \sum_{n=1}^{\infty} C_n \, e^{(n+1/2)\xi} P_n^1(\cos \eta) \sin \eta. \tag{22}$$

The interface conditions require that

$$A_n = C_n, \qquad \mu_1 B_n = \mu_2 C_n. \tag{23}$$

If $\mu = \mu_2/\mu_1$, then $A_n = C_n$, $B_n = \mu C_n$.

Now on $\xi = \alpha$

$$\frac{\sin \eta}{(\cosh \alpha - \cos \eta)^{3/2}} = 2\sqrt{2} \sum_{n=1}^{\infty} e^{-(n+1/2)\alpha} P_n^1(\cos \eta). \tag{24}$$

It follows that the boundary condition on the sphere is satisfied if

$$C_n = \frac{2\sqrt{2}\, e^{-(n+1/2)\alpha}}{\cosh\left(n + \frac{1}{2}\right)\alpha + \mu \sinh\left(n + \frac{1}{2}\right)\alpha}. \tag{25}$$

The coefficients A_n, B_n are then found from [23]. It is also easily verified that the velocities $\mathbf{q}_j$, $j = 1, 2$ vanish at infinity.

The torque on the rotating sphere can be shown to be $-T\hat{k}$ where

$$T = 4\sqrt{2}\, \pi \mu \omega a^3 \sinh^3 \alpha \sum_{n=1}^{\infty} n(n+1)(A_n + B_n). \tag{26}$$

with a denoting the radius of the sphere.

Now

$$A_n + B_n = \frac{2\sqrt{2}(1+\mu)\, e^{-(n+1/2)\alpha}}{\cosh\left(n + \frac{1}{2}\right)\alpha + \mu \sinh\left(n + \frac{1}{2}\right)\alpha} = \frac{4\sqrt{2}(1+\mu)\, e^{-(2n+1)\alpha}}{1 + \mu + (1-\mu)\, e^{-(2n+1)\alpha}}. \tag{27}$$

Hence

$$\tau = \frac{T}{8\pi\mu\omega a^3} = 4(1+\mu) \sinh^3 \alpha \sum_{n=1}^{\infty} \frac{n(n+1)\, e^{-(2n+1)\alpha}}{\{1 + \mu + (1-\mu)\, e^{-(2n+1)\alpha}} = 4 \sinh^3 \alpha \sum_{n=1}^{\infty} \frac{n(n+1)\, e^{-(2n+1)\alpha}}{1 + \lambda\, e^{-(2n+1)\alpha}} \tag{28}$$

Table 1.

α	d/a	μ=0 τ	μ=0.1 τ	μ=0.2 τ	μ=0.5 τ	μ=1.0 τ
5.0	74.2099	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	27.3082	1.0000	1.0000	1.0000	1.0000	1.0000
3.0	10.0677	0.9999	0.9999	0.9999	1.0000	1.0000
2.0	3.7622	0.9977	0.9981	0.9984	0.9992	1.0000
1.0	1.5431	0.9675	0.9732	0.9780	0.9888	1.0000
0.5	1.1276	0.9250	0.9371	0.9476	0.9725	1.0000
0.4	1.0811	0.9172	0.9302	0.9417	0.9691	1.0000
0.3	1.0453	0.9106	0.9244	0.9367	0.9662	1.0000
0.2	1.0201	0.9056	0.9201	0.9329	0.9639	1.0000
0.1	1.0050	0.9026	0.9173	0.9305	0.9624	1.0000
α	d/a	μ=2.0 τ	μ=5.0 τ	μ=10.0 τ	μ=20.0 τ	μ=50.0 τ
5.0	74.2099	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	27.3082	1.0000	1.0000	1.0000	1.0000	1.0000
3.0	10.0677	1.0000	1.0001	1.0001	1.0001	1.0001
2.0	3.7622	1.0008	1.0016	1.0019	1.0021	1.0023
1.0	1.5431	1.0115	1.0234	1.0290	1.0322	1.0343
0.5	1.1276	1.0308	1.0657	1.0831	1.0936	1.1006
0.4	1.0811	1.0354	1.0768	1.0981	1.1112	1.1110
0.3	1.0453	1.0397	1.0878	1.1134	1.1295	1.1405
0.2	1.0201	1.0433	1.0975	1.1277	1.1471	1.1608
0.1	1.0050	1.0456	1.1045	1.1386	1.1615	1.1782

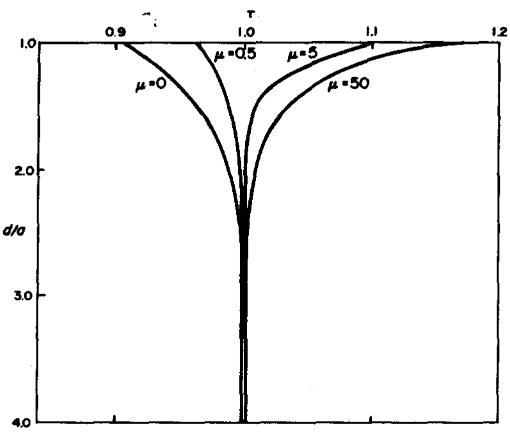

Figure 1.

where $\lambda = (1-\mu)/(1+\mu)$. Thus τ may be written as

$$\tau = 4\sinh^3\alpha \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e^{-(2n+1)\alpha}(-1)^m\lambda^m\, e^{-(2n+1)m\alpha} \quad [29]$$

$$= 4\sinh^3\alpha \sum_{m=0}^{\infty}(-1)^m\lambda^m \sum_{n=1}^{\infty} n(n+1)\, e^{-(2n+1)(m+1)\alpha} \quad [30]$$

so that

$$\tau = \sinh^3\alpha \sum_{m=0}^{\infty}(-1)^m\lambda^m \operatorname{cosech}^3(m+1)\alpha. \quad [31]$$

The following limiting cases are of interest

(i) $\alpha \to 0$,
$$\tau \to \sum_{m=0}^{\infty}\frac{(-\lambda)^m}{(m+1)^3} \quad [32]$$

(ii) $\mu \to 0$,
$$\tau \to \sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)^3} = \frac{3}{4}\zeta(3) = 0.901543 \quad [33]$$

(iii) $\mu \to \infty$,
$$\tau \to \sum_{m=0}^{\infty}\frac{1}{(m+1)^3} = \zeta(3) = 1.20206. \quad [34]$$

A table of numerical values for the torque is given. It is remarked that the more general problem of two spheres one in each phase rotating with unequal angular velocities can be treated by the same method.

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